Deciding whether a Grid is a Topological Subgraph of a Planar Graph is NP-Complete

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Abstract

We study the computational complexity of the following decision problem: given a positive integer k and a planar graph G, is the $k \times k$ grid a topological subgraph of G? We prove that this problem is NP-complete, even when restricted to planar graphs of maximum degree six, via a reduction from the PLANAR MONOTONE 3-SAT PROBLEM.

1 Introduction

Given a graph G, the *subdivision* of an edge uv of G consists of its deletion and the addition of a new path of length two with ends u and v. For graphs G and H, we say that H is a *topological subgraph* of G, or that G *contains a subdivision* of H, if G has a subgraph isomorphic to a graph obtained from H by repeatedly subdividing edges. This notion appears for example in the classical characterization of planar graphs by Kuratowski.

Our work concerns the computational complexity of the GRID TOPOLOGICAL SUBGRAPH CONTAINMENT (GRID TSC) PROBLEM in planar graphs. The general question, known as the TOPOLOGICAL SUBGRAPH CONTAINMENT (TSC) PROBLEM or as the SUBGRAPH HOMEOMORPHISM PROBLEM, is to determine for two given graphs G and H, whether H is a topological subgraph of G. To the best of our knowledge, investigations on this problem started with the work of LaPaugh et al. [10], and related results can be found in [1, 5, 6, 8, 13, 14]. On the one hand, an algorithmic implication of the famous Graph Minor Theorem of Robertson and Seymour [11] is that, when H is fixed, the TSC PROBLEM can be solved in time polynomial in the order of the input graph G. More recently, Grohe et al. [8] showed that the TSC PROBLEM is fixed-parameter tractable by the order of H. On the other hand, when H is part of the input, the problem is NP-complete even when restricted to planar input graphs. Indeed, when G is a planar graph on n vertices and H is a cycle on n vertices, then the HAMILTON CYCLE PROBLEM in planar graphs is obtained, which is known to be NP-complete [7].

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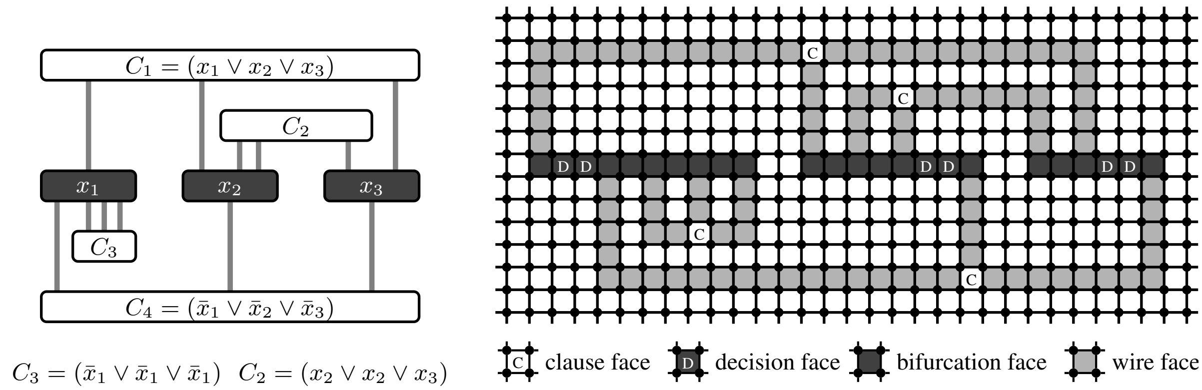


a) A monotone rectilinear representation of $\phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$.

b) Plan for adding gadgets to the grid $\tilde{G}$. Variable gadgets consist of decision and bifurcation gadgets.

Figure 1

Grids are fundamental structures in graph algorithms due to the result of Robertson and Seymour [12], within the proof of the Graph Minor Theorem, which roughly claims that graphs of large treewidth necessarily contain large grid minors. In recent years, researchers have developed constant-factor approximation algorithms for finding largest grid minors in graphs excluding a fixed minor [3] as well as in planar graphs [9].

In this work, we prove that finding largest topological grid minors in planar graphs is NP-hard. More precisely, we study the GRID TSC PROBLEM, which is to decide whether a given graph G contains the $k \times k$ grid as a topological subgraph, where k is part of the input.

Theorem 1. *The* GRID TSC PROBLEM *in planar graphs is NP-complete, even when restricted to planar graphs with maximum degree 6.*

Our proof of the previous theorem is a reduction, sketched in Section 1.1, from the PLANAR MONOTONE 3-SAT PROBLEM, which was shown to be NP-complete by de Berg and Khosravi [2].

Regarding our theorem, we think that, using similar ideas (though, it is not straightforward), it is possible to improve the maximum degree to 4, which would be best possible. We believe that our result and the reduction strategy will help in resolving the computational complexity of the GRID MINOR CONTAINMENT PROBLEM in planar graphs, is still open.

1.1 Reduction Idea

Consider an instance of the PLANAR MONOTONE 3-SAT PROBLEM, i.e., a drawing $\mathcal{R}$ as in Figure 1a). Denote by $\phi = \phi(\mathcal{U}, \mathcal{C})$ the 3-SAT formula corresponding to $\mathcal{R}$. To prove that the GRID TSC PROBLEM in planar graphs is NP-hard (Theorem 1), we construct a planar graph G_ϕ and define a value k such that the $k \times k$ grid is a topological subgraph of G_ϕ if and only if ϕ is satisfiable.

The construction of G_ϕ starts with the $k \times k$ grid $\tilde{G}$. We present gadgets for clauses and variables as well as wire gadgets that are used to connect variable gadgets to clause gadgets. Each variable gadget consists of several bifurcation gadgets and one decision gadget. The purpose of the decision gadget is to encode whether a variable of ϕ is set to TRUE or FALSE and the purpose of a clause gadget is to ensure that at most two of the three literals of the clause are set to FALSE. The bifurcation and wire gadgets duplicate and propagate the information throughout the graph G_ϕ. Using the drawing $\mathcal{R}$, these gadgets are placed into some *inner* faces of $\tilde{G}$, i.e., faces of $\tilde{G}$ that are bounded by a cycle of length 4, see Figure 1.

We show that, if G_ϕ contains a subdivision H of the $k \times k$ grid, then, roughly speaking, H is $\tilde{G}$ except for a few local differences. This is made more precise later on by introducing the concept of an X-normal subdivision. In the construction of G_ϕ, for each variable gadget, an edge of $\tilde{G}$ is deleted. Such a deletion forces the corresponding path of H to bend and, as a result, all paths along some wire gadgets connecting that variable to positive or negative clauses are forced to bend. Here, a bend of a path can be interpreted as a variable pushing the value FALSE towards a clause. The clause gadget, placed into an inner face f of $\tilde{G}$, is designed in such a way that at most two of the paths (originally in $\tilde{G}$) using an edge on the boundary of f can bend into the face f. As a consequence, if the three variables of the clause are set to FALSE, one of the paths cannot bend into the face f and G_ϕ does not contain a subdivision of the $k \times k$ grid.

2 Preliminaries

The Planar Monotone 3-SAT Problem

Consider a set $\mathcal{U} = \{x_1, \ldots, x_n\}$ of boolean variables and a set $\mathcal{C} = \{C_1, \ldots, C_m\}$ of clauses over $\mathcal{U}$, where each clause C_i with $i \in \{1, \ldots, m\}$ is a disjunction of at most 3 literals, i.e., variables from $\mathcal{U}$ or their negation. Then, $\phi = \phi(\mathcal{U}, \mathcal{C}) = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ is a *3-SAT formula* over $\mathcal{U}$ and is called *satisfiable* if there exists an assignment of TRUE and FALSE to the variables in $\mathcal{U}$ such that ϕ evaluates to TRUE. If a clause C contains only positive or only negative literals, then C is called *positive* or *negative*, respectively. A 3-SAT formula ϕ is called *monotone* if each clause in ϕ is either positive or negative. Moreover, ϕ is called *planar* if the following bipartite graph G is planar: the vertex set of G is $\{x_1, \ldots, x_n\} \cup \{C_1, \ldots, C_m\}$ and $\{x, C\}$ is an edge of G if and only if the clause C uses x or its negation $\bar{x}$.

In [2], de Berg and Khosravi introduce monotone rectilinear representations, which combine the properties monotone and planar. Assume that $\phi = \phi(\mathcal{U}, \mathcal{C})$ is a monotone and planar 3-SAT formula. Consider an orthogonal coordinate system in the plane consisting of a horizontal and a vertical axis. A *monotone rectilinear representation* of ϕ is a drawing in the plane with the following properties, see Figure 1a):

- Variables in $\mathcal{U}$ and clauses in $\mathcal{C}$ are represented by pairwise disjoint rectangles in the plane, each of whose sides is parallel to the horizontal or the vertical axis.
- The horizontal axis intersects each rectangle representing a variable in $\mathcal{U}$ and no rectangle representing a clause in $\mathcal{C}$. Further, each rectangle representing a positive clause in $\mathcal{C}$ is drawn above the horizontal axis and each rectangle representing a negative clause in $\mathcal{C}$ is drawn below the horizontal axis.
- For each variable $x \in \mathcal{U}$ and for each clause $C \in \mathcal{C}$ such that C contains x or $\bar{x}$, there is a vertical

line segment that joins the rectangles representing x and C and that does neither intersect other line segments nor other rectangles.

Given a monotone rectilinear representation of a 3-SAT formula ϕ, the PLANAR MONOTONE 3-SAT PROBLEM is to decide whether ϕ is satisfiable. The following holds.

Theorem 2 (de Berg and Khosravi, Theorem 1 in [2]). *The* PLANAR MONOTONE 3-SAT PROBLEM *is NP-complete.*

Consider a planar monotone 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$ together with a monotone rectilinear representation $\mathcal{R}$. Throughout this paper, we do not distinguish between variables of ϕ and their representations as a rectangles in $\mathcal{R}$, and similarly for clauses. Furthermore, without loss of generality, we assume that each variable in $\mathcal{U}$ appears in at least one positive clause and at least one negative clause, i.e., ϕ uses both literals x and $\bar{x}$ for each $x \in \mathcal{U}$. Also, we assume that each clause contains exactly three (not necessarily distinct) literals and, in $\mathcal{R}$, each clause is incident to exactly three vertical lines. Moreover, we assume that, for each variable x, the line segments joining x to positive clauses touch x in its left half and line segments joining x to negative clauses touch x in its right half.

Definitions and terminology

Consider two graphs H and G. Recall that H is a subdivision of G, if H can be obtained from G by repeatedly subdividing edges. We say that two paths P and Q are *internally vertex-disjoint* if every vertex $v \in V(P)$ with $\deg_P(v) = 2$ satisfies $v \notin V(Q)$ and vice versa. If H is isomorphic to a subdivision of G and G does not have any isolated vertices, then there are two injective maps

$$f_V : V(G) \to V(H) \quad \text{and} \quad f_E : E(G) \to \{P \subseteq H\colon\ P \text{ is a path in } H\}$$

such that, for all $\{u, v\} \in E(G)$, the path $f_E(\{u, v\})$ has end vertices $f_V(u)$ and $f_V(v)$ as well as that the paths $f_E(e)$ and $f_E(e')$ are internally vertex-disjoint for all distinct $e, e' \in E(G)$. In the following, any such maps f_V and f_E are called *vertex and edge maps for* H.

Throughout this paper, let $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$. For each $k \geq 3$, the $k \times k$ grid, denoted by $\tilde{G}_k$, is the graph with vertex set $\tilde{V}_k := \{(i, j)\colon\ i, j \in [k]\}$ and edge set

$$\tilde{E}_k := \left\{\{(i, j), (i', j')\}\colon\ |i - i'| + |j - j'| = 1\right\}.$$

Fix some $k \geq 3$ and let $\tilde{G} := \tilde{G}_k$. The *canonical embedding* of $\tilde{G}$ refers to a drawing of $\tilde{G}$ in the plane, where the vertex $(i, j) \in V(\tilde{G})$ is embedded at the point (i, j) in a coordinate system whose horizontal axis refers to the first coordinate and whose vertical axis refers to the second coordinate and each edge of $\tilde{G}$ is represented by a line segment. The unique infinite face of the canonical embedding of $\tilde{G}$ is called the *outer face* of $\tilde{G}$ and all other faces of $\tilde{G}$ are referred to as *inner faces* of $\tilde{G}$. For an inner face f of $\tilde{G}$, the terms *left/right/top/bottom edge* as well as the terms *face directly left/right/below/above* are defined in the obvious way according to the canonical embedding of $\tilde{G}$.

For $i \in [k]$, the path that is induced in $\tilde{G}$ by the vertices in $\{(i, j)\colon\ j \in [k]\}$ is called the i^{th} *vertical path* of $\tilde{G}$ and, for $j \in [k]$, the path that is induced in $\tilde{G}$ by the vertices in $\{(i, j)\colon\ i \in [k]\}$ is called the j^{th} *horizontal path* of $\tilde{G}$. Edges of $\tilde{G}$ are called either *vertical* or *horizontal*, accordingly.

The notions defined above for grids naturally extend, through the vertex and edge maps, to subdivisions of grids. Let H be a subdivision of $\tilde{G}$ and let f_V and f_E be vertex and edge maps for H. The *outer face* of H is the unique face of H whose boundary does not contain a vertex of degree 4. For $i \in [k]$, let $P_i^v \subseteq H$ be the unique $f_V((i,1))$,$f_V((i,k))$-path obtained by glueing the paths $f_E(e)$ for all $e \in E(\tilde{G})$ that belong to the i^{th} vertical path of $\tilde{G}$ together and let $P_i^h \subseteq H$ be the unique $f_V((1,i))$,$f_V((k,i))$-path obtained by glueing the paths $f_E(e)$ for all $e \in \tilde{G}$ that belong to the i^{th} horizontal path of $\tilde{G}$ together. Clearly, the paths $P_1^v, \ldots, P_k^v$ are pairwise vertex-disjoint and the paths $P_1^h, \ldots, P_k^h$ are also pairwise vertex-disjoint. The paths P_i^v, P_j^h with $i, j \in [k]$ are called *grid-paths* of H. By the *type* of a grid-path, we refer to the property of the grid-path being horizontal or vertical. For all $i, j \in [k]$, the unique common vertex of the paths P_i^v and P_j^h, namely $f_V((i,j))$, is called an *intersection vertex* of H.

Observation 3. *Let H be a subdivision of a $k \times k$ grid and let P and P' be two distinct grid-paths of H. Then $|V(P) \cap V(P')| \leq 1$. Moreover, $|V(P) \cap V(P')| = 0$ if and only if P and P' are of the same type, and $|V(P) \cap V(P')| = 1$ if and only if P and P' are of distinct types. In particular, P and P' do not have a common edge.*

A planar graph G *is uniquely embeddable* if all embeddings of G in the plane define the same set of facial boundaries, see [4] for details. Clearly, if a planar graph G is uniquely embeddable, then also any subdivision of G is uniquely embeddable. The following proposition is obtained by the fact that the $k \times k$ grid for $k \geq 3$ is a subdivision of a 3-connected graph and applying Whitney's Theorem (see Theorem 4.3.2 in [4]).

Proposition 4. *For each $k \geq 3$, the $k \times k$ grid $\tilde{G}$ and every subdivision of $\tilde{G}$, are uniquely embeddable.*

The following observation is a direct consequence of the previous proposition.

Observation 5. *Consider a planar graph G and a subgraph $H \subseteq G$ that is isomorphic to a subdivision of the $k \times k$ grid for some $k \geq 3$. Let P and Q be a horizontal and a vertical path of G, respectively, and let v be their unique common vertex. In every embedding of G, the paths P and Q cross each other at v, i.e., a small circle around v touches P and Q alternately.*

3 Gadgets

In this section, we consider the $k \times k$ grid $\tilde{G}$ for some large $k \in \mathbb{N}$ along with its canonical embedding. Let B_{outer} be the boundary of the outer face of $\tilde{G}$. Throughout this work, we refer to deleting an edge, subdividing an edge, adding a new vertex, and adding a new edge as *modifications*. Let G be a plane graph obtained from $\tilde{G}$ by modifications. Clearly, G contains all vertices of $\tilde{G}$ and we refer to each such vertex as an *original grid vertex*.

For $X \subseteq V(\tilde{G})$, a graph $H \subseteq G$ that is isomorphic to a subdivision of $\tilde{G}$ is called X-*normal* if B_{outer} bounds the outer face of H and a vertex map f_V for H can be chosen such that $f_V(w) = w$ for all $w \in V(\tilde{G}) \setminus X$. Every time we consider an X-normal subdivision, we implicitly fix such a choice of f_V. Observation 6 follows.

Observation 6. *Let G be a graph obtained from $\tilde{G}$ by modifications and let $H \subseteq G$ be an X-normal subdivision for some $X \subseteq V(\tilde{G})$.*

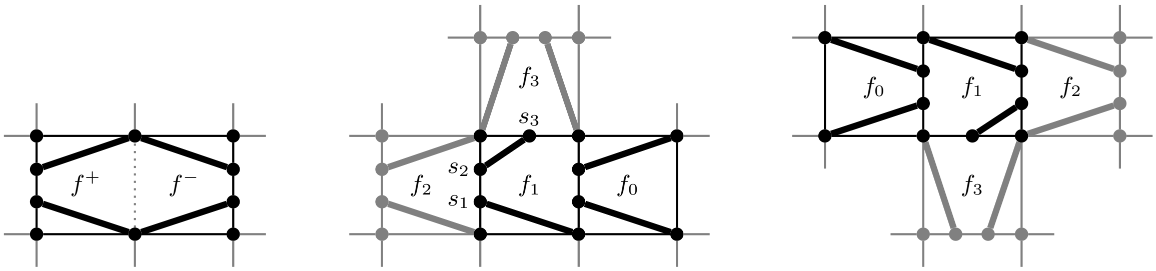
- *Each vertex $(i,j) \in V(\tilde{G}) \setminus X$ is the unique common vertex of the i^{th} vertical grid-path of H and the j^{th} horizontal grid-path of H.*
- *For each $i \in [k]$, if the i^{th} vertical grid-path of H uses an original grid vertex $w \neq (i,j)$ for all $j \in [k]$, then $w \in X$. Similarly, for each $j \in [k]$, if the j^{th} horizontal grid-path of H uses an original grid vertex $w \neq (i,j)$ for all $i \in [k]$, then $w \in X$.*

In order to construct the gadgets, we apply modifications to certain inner faces of $\tilde{G}$. These modifications often split an inner face of $\tilde{G}$ into several faces, or simply change the boundary of an inner face. In this work, for a graph G obtained from $\tilde{G}$ by modifications, by abuse of notation, we refer to an inner face of G as the inner face of $\tilde{G}$ with the updated boundary if necessary. Further, for a face f of G, we say that a modification is *outside* f if no edge on the boundary of f is deleted and neither new edges nor new vertices are embedded inside f.

In Section 1.1, we informally used the idea of "paths that are forced to bend in order to have a subdivision of a grid". Next, we formalize this idea, which is a crucial concept throughout the paper. For $d \in \mathbb{N}$, an f_0,f_d-face-sequence in $\tilde{G}$ is an alternating sequence $(f_0, e_1, f_1, \ldots, e_d, f_d)$ of distinct faces and edges of $\tilde{G}$ such that e_h bounds the faces f_{h-1} and f_h for all $h \in [d]$. Let G be a plane graph obtained from $\tilde{G}$ by modifications. If an edge $e = \{x, y\} \in E(\tilde{G})$ is in G or it has been subdivided in order to obtain G, we use P_e to denote the x,y-path in G that replaces e and we assume that P_e is embedded exactly where e was embedded. If $e = \{x, y\} \in E(\tilde{G})$ has been deleted, we define $V(P_e) = \{x, y\}$ and $E(P_e) = \emptyset$. Consider an X-normal subdivision $H \subseteq G$ for some $X \subseteq V(\tilde{G})$. Denote by f_V and f_E a vertex map and an edge map for H. For an arbitrary integer $d \geq 1$, let $F := (f_0, e_1, f_1, \ldots, e_d, f_d)$ be a face-sequence in $\tilde{G}$ and let e_0 be the edge on the boundary of f_0 satisfying $e_0 \cap e_1 = \emptyset$. We say that H *relaxes* along F, if $f_E(e_h) \subseteq P_{e_h}$ for all $h \in [d]$ and that H *pushes* along F if the path $f_E(e_{h-1})$ uses at least one vertex in $V(P_{e_h}) \setminus V(\tilde{G})$ for all $h \in [d]$. Now, consider the set $\mathscr{H}$ of graphs $H \subseteq G$ such that H is an X-normal subdivision and it pushes along (f_0, e_1, f_1). If $\mathscr{H} \neq \emptyset$ and every graph $H \in \mathscr{H}$ pushes along F, then we say that (G, X) *forces to push along* F. In the case that $\mathscr{H} = \emptyset$, we say that (G, X) *disqualifies*.

A common modification when constructing gadgets will be to add an arrow to some inner face f of $\tilde{G}$. To define this, let $i, j \in [k]$ be such that f is bounded by the cycle $((i,j), (i+1,j), (i+1,j+1), (i,j+1))$. To add a *left arrow* in f means to replace the edge $\{(i,j), (i,j+1)\}$ by the path $((i,j), s_1, s_2, (i,j+1))$, where s_1 and s_2 are new vertices, and to add the edges $\{(i+1,j), s_1\}$ and $\{(i+1,j+1), s_2\}$. Similarly, we define a *right arrow*, an *up arrow*, and a *down arrow*.

When presenting the gadgets in the following subsections, we always assume that the gadgets are placed close to the middle of $\tilde{G}$. In particular, we assume that all modified faces are inner faces that do not contain a vertex of B_{outer} in their boundary. In the figures showing gadgets, new edges are drawn thicker than edges in $\tilde{G}$ and edges arising from subdivisions. Vertices and edges that do not belong to the gadget subgraph are colored gray.



a) A decision gadget. **b)** A positive bifurcation gadget. **c)** A negative bifurcation gadget.

Figure 2: Decision and bifurcation gadgets.

3.1 Variable Gadgets

The variable gadget consists of one decision gadget and several bifurcation gadgets. The decision gadget is the part of the variable gadgets that encodes the assignment of TRUE or FALSE to the variable. The bifurcation gadget replicates the information encoded by the decision gadget as many times as needed, i.e., once for each clause using the variable.

Decision gadget

Consider a vertical edge e of $\tilde{G}$ that does not belong to B_{outer}. Denote by f^+ and f^- the two inner faces of $\tilde{G}$ whose right edge is e and whose left edge is e, respectively. To add a decision gadget for $x \in \mathcal{U}$ means to add a left arrow in f^+, to add a right arrow in f^-, and to delete the edge $e_x = e$, see Figure 2a). Faces f^+ and f^-, are called the *positive* and the *negative face* of the variable x, respectively, and the new edges in f^+ and f^- are called the *positive* and the *negative edges* of the variable x, respectively.

Bifurcation gadget

Roughly speaking, a *positive bifurcation gadget* consists of two left arrows, where one of them is slightly twisted, see Figure 2b). More precisely, consider two inner faces f_0 and f_1 of $\tilde{G}$ such that f_1 is directly left of f_0. Denote by e_1 the unique edge of $\tilde{G}$ that is on the boundary of f_0 and f_1. Let e_2 and e_3 be the left edge and the top edge of f_1, respectively. The following modifications are applied to add a positive bifurcation gadget in f_0 and f_1. Add a left arrow in f_0. Subdivide e_2 with two vertices s_1 and s_2 and subdivide e_3 once, say with the vertex s_3. Without loss of generality, assume that s_2 and s_3 have a common neighbor. Insert the new edge $\{s_2, s_3\}$ and a new edge joining s_1 to the common vertex of the bottom edge and the right edge of f_1. Faces f_1 and f_0 are called *bifurcation faces*, and f_0 is also called the *right connection* face of the positive bifurcation gadget. Moreover, the inner face f_2 of $\tilde{G}$, which is directly left of f_1, is called the *left connection face* and the inner face f_3 of $\tilde{G}$, which is directly above f_1, is called the *top connection face* of the positive bifurcation gadget. The faces f_2 and f_3 do not belong to the positive bifurcation gadget but are used to connect it to other gadgets are indicated in Figure 2b).
The *negative bifurcation gadget* is obtained by rotating the positive bifurcation gadget around 180 de-

grees, see Figure 2c) and the terminology for the negative bifurcation gadget is naturally adapted. Observe, that the negative bifurcation gadget defines a left, a right and a bottom connection face but only the left connection face belongs to the gadget itself.

3.1.1 Assembling the Variable Gadget

In what follows, we consider an instance of PLANAR MONOTONE 3-SAT, i.e., a monotone rectilinear representation of a 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$. Fix a variable $x \in \mathcal{U}$. We denote by $\deg^+(x)$ (resp. $\deg^-(x)$) the number of appearances of x (resp. $\bar{x}$) in the clauses in $\mathcal{C}$. A *variable gadget* for x consists of $\deg^+(x)$ positive bifurcation gadgets and $\deg^-(x)$ negative bifurcation gadgets, one left connection face associated to a positive bifurcation gadget and one right connection face associated to a negative bifurcation gadget; the extended variable gadget additionally consists of all top and bottom connection faces associated to these bifurcation gadgets, see Figure 3. More precisely, let $(f_1, e_2, \ldots, e_d, f_d)$ be a face-sequence with $d := 2(\deg^+(x) + \deg^-(x))$ such that f_{h+1} is the face directly right of f_h for all $h \in [d-1]$. For each $h \in [\deg^+(x)]$, the faces f_{2h-1} and f_{2h} are modified according to the positive bifurcation gadget, the edge $e_{2\deg^+(x)+1}$ is deleted, and, for each integer h with $\deg^+(x) + 1 \leq h \leq \deg^+(x) + \deg^-(x)$, the faces f_{2h-1} and f_{2h} are modified according to the negative bifurcation gadget. Note that the faces $f_{2\deg^+(x)}$ and $f_{2\deg^+(x)+1}$ are automatically modified according to a decision gadget, which we refer to as the decision gadget for x. Moreover, the following modifications are applied to the face f_0 directly left of f_1 and the face f_{d+1} directly right of f_d, which are, roughly speaking, the left connection face of the first positive bifurcation gadget and the right connection face of the last negative bifurcation gadget. A new vertex v_0, which is adjacent to both ends of the right edge of f_0, is inserted in f_0 and a new vertex v_{d+1}, which is adjacent to both ends of the left edge of f_{d+1}, is inserted f_{d+1}. Naturally, those faces are now called the *left and the right connection face* of the variable gadget for x, respectively. Again, see Figure 3. To understand how the variable gadget works, we define an *extended variable gadget*, with the following additional modifications: an up arrow is added to each top connection face of a positive bifurcation gadget and a down arrow is added to each bottom connection face of a negative bifurcation gadget. These faces are now referred to as the *top and bottom connection faces* of the variable gadget for x, respectively.

Consider a graph G obtained from $\tilde{G}$ by adding one extended variable gadget corresponding to a variable $x \in \mathcal{U}$ and possibly further modifications outside the faces belonging to the extended variable gadget. Denote by f^+ and f^- the positive and the negative face of the decision gadget that is part of the variable gadget. Let $\mathcal{F}^+$ be the family of face-sequences that, for each face f, where f is a top connection face or the left connection face, contains the f^+,f-face-sequence,   that contains only the faces f, f^+, and positive bifurcation faces. Similarly, let $\mathcal{F}^-$ be the family of face-sequences that, for each face f, where f is a bottom connection face or the right connection face, contains the f^-,f-face-sequence, that contains only the faces f^-, f and negative bifurcation faces. Moreover, let e_x be the unique edge of $\tilde{G}$ that was deleted due to the variable gadget for x, i.e., the unique edge between f^+ and f^-.

Consider an X-normal subdivision $H \subseteq G$ for some set $X \subseteq V(\tilde{G})$, where X contains no vertex of the extended variable gadget. The grid-path of H that contains both end vertices of e_x is called the *decision*

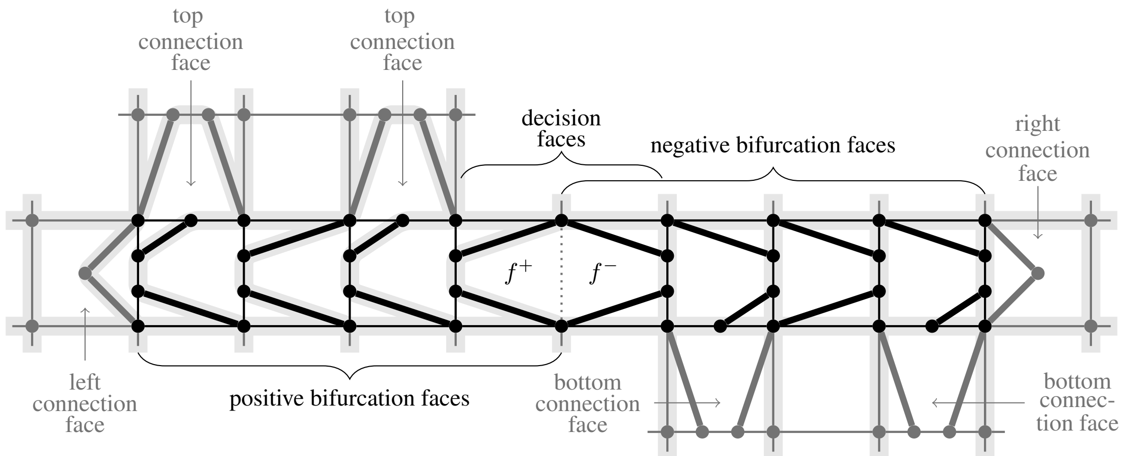


Figure 3: The variable gadget. The corresponding part of a "negative subdivision" is highlighted.

path of x. The next two claims follow from the structure of variable gadgets and Observation 6.

Claim 7. *Let G be a graph obtained from $\tilde{G}$ by adding an extended variable gadget for $x \in \mathcal{U}$ and by further modifications outside the faces of the extended variable gadget for x. Denote by $\mathcal{F}^+$ and $\mathcal{F}^-$ the face-sequences defined above. Consider a set $X \subseteq V(\tilde{G})$ containing no vertex of the extended variable gadget for x.*

a) For every X-normal subdivision $H \subseteq G$, the decision path of x in H uses either both positive edges or both negative edges of x.
b) The pair (G, X) forces to push along each face-sequence in $\mathcal{F}^+$ or (G, X) disqualifies.
c) The pair (G, X) forces to push along each face-sequence in $\mathcal{F}^-$ or (G, X) disqualifies.

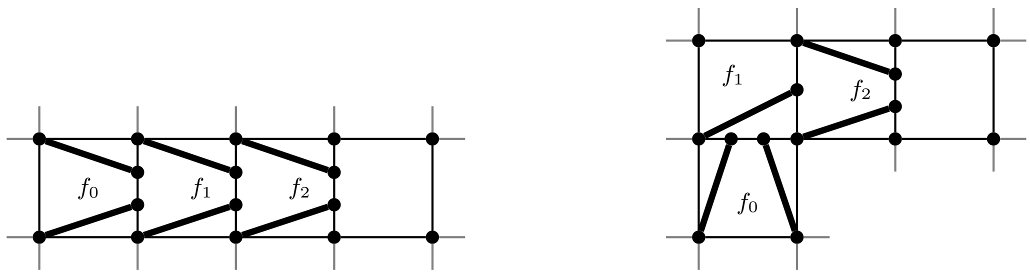
Consider the setting of Claim 7. We denote by G^+ the plane graph obtained from G by applying the following modifications. For each face f' directly above a top connection face f of the variable gadget, add a new vertex v in f' that is adjacent to the two original grid vertices that are on the boundary of f and f'.

Claim 8. *If the graph obtained from G by reinserting the edge e_x contains an X-normal subdivision, then G^+ contains an X-normal subdivision, where the decision path of x uses both positive edges of x and all edges in $E(G^+) \setminus E(G)$.*

Similarly, define a graph G^-, where vertices are added to each face directly below a bottom connection face. Claim 8 holds accordingly for G^-.

3.2 Wire Gadgets

The purpose of the wire gadgets is to transfer the information from the variable gadgets to the clause gadgets. Before presenting them, some further notation is introduced. Let $F = (f_0, e_1, f_1, \ldots, e_d, f_d)$ be a face-sequence for some integer d. Then, F is called *straight* if $d \leq 1$ or, $e_h \cap e_{h+1} = \emptyset$ for



a) A straight wire. **b)** An almost straight wire (turning at f_1).

Figure 4: Wire gadgets along $F = (f_0, e_1, f_1, e_2, f_2)$.

all $h \in [d-1]$. Moreover, for an integer $h \in [d-1]$, the face-sequence F *is almost straight (and it turns at f_h)* if F is not straight but $F' := (f_0, e_1, \ldots, f_h)$ and $F'' := (f_h, e_{h+1}, \ldots, f_d)$ are straight. Let $F = (f_0, e_1, f_1, \ldots, e_d, f_d)$ be a face-sequence that is straight or almost straight. To add a *wire gadget* along F consists of the following modifications: add an arrow to each face of F in such a way that the arrows point along F, see Figure 4. If F turns at f_h, then only one edge of the arrow is added to f_h, as shown in Figure 4b). The next two claims follow from the structure of wire gadgets and Observation 6.

Claim 9. *Let $d \geq 2$ and let $F = (f_0, e_1, \ldots, e_d, f_d)$ be a face-sequence in $\tilde{G}$ that is straight or almost straight. Consider a graph G that is obtained from $\tilde{G}$ by adding a wire gadget along F and further modifications outside the faces in F. Let $X \subseteq V(G)$ be a set that contains no vertex of a face in F. Then, (G, X) forces to push along F or (G, X) disqualifies.*

Consider the setting of Claim 9. Denote by e_0 the edge on the boundary of f_0 with $e_0 \cap e_1 = \emptyset$ and by e_{d+1} the edge on the boundary of f_d with $e_{d+1} \cap e_d = \emptyset$. Let G_1 (resp. G_2) be the graph obtained from G by inserting a new vertex that is adjacent to the end vertices of e_0 (resp. e_{d+1}). The following claim holds.

Claim 10. *If G_1 has an X-normal subdivision containing the two edges in $E(G_1) \setminus E(G)$, then G_2 has an X-normal subdivision containing the two edges in $E(G_2) \setminus E(G)$.*

3.3 Clause Gadgets

Consider a positive clause $C \in \mathcal{C}$. A *positive clause gadget* for C consists of a *clause face* f, which is an inner face of $\tilde{G}$, and three *connection faces*, which are the faces directly left, below, and right of f, respectively. Figure 6 depicts the positive clause gadget. With reference to the vertex names in Figure 6, define $X_C = \{g_2, g_3, g_6, g_7\}$. The following claim states a crucial property of the positive clause gadget.

Claim 11. *Fix an inner face f of $\tilde{G}$ and denote by F the set of all inner faces of $\tilde{G}$ whose boundary contains a vertex on the boundary of f. Consider a graph G obtained from $\tilde{G}$ by adding a positive clause gadget with inner face f and further modifications outside the faces in F. Let $X \subseteq V(G)$ be a set that contains X_C and such that $X \setminus X_C$ contains no vertex that is on the boundary of a face in F.*

There is no X-normal subdivision $H \subseteq G$ that simultaneously pushes along (f_h, e_h, f) for each connection face f_h of the clause gadget, where e_h denotes the unique edge of $\tilde{G}$ that is on the boundary of f_h and f.

Proof. Let G, f, F, and X be as in the statement. Denote by f_l, f_b, and f_r the faces directly left, below and right of f, respectively, and for each $h \in \{l, b, r\}$, denote by e_h the unique edge of $\tilde{G}$ that is on the boundary of f and on the boundary of f_h. In the following, we use the vertex names as in Figure 6. Choose $i, j \in [k]$ such that $g_6 = (i, j)$.

Assume for a contradiction that there is an X-normal subdivision $H \subseteq G$ that, for all $h \in \{l, b, r\}$, pushes along (f_h, e_h, f). Denote by $P_1^v, \ldots, P_k^v$ and $P_1^h, \ldots, P_k^h$ the vertical and horizontal paths of H. Let V^* be the set of vertices that are embedded in a face in F or in the boundary of a face in F. Using only vertices from V^*,

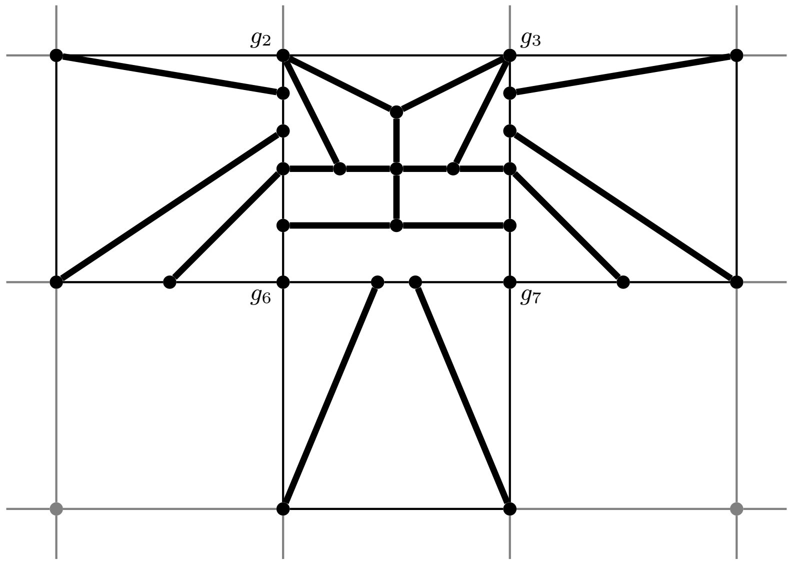
- the path P_i^v joins $g_9 = (i, j-1)$ and $(i, j+2)$,
- the path P_{i+1}^v joins $g_{10} = (i+1, j-1)$ and $(i+1, j+2)$, and
- the path P_j^h joins $g_5 = (i-1, j)$ and $g_8 = (i+2, j)$.

The drawing of G contains a face f' bounded by the paths (g_1, s_1, s_2, g_5) and $P_{\{g_1, g_5\}}$. As H pushes along (f_l, e_l, f), the path $P_{\{g_1, g_5\}}$ and the segment of the path P_{i-1}^v that joins g_1 to g_5 forms a cycle C, such that the face f' lies completely on one side of C. Since distinct vertical paths do not intersect, the paths P_i^v and P_{i+1}^v cannot use any vertex on the boundary of f'. In particular, P_i^v uses neither s_1 nor s_2. Similarly, it follows that P_j^h uses neither s_3 nor s_4, and that P_{i+1}^v uses neither s_5 nor s_6. As P_{i-1}^v intersects with P_j^h in $g_5 = (i-1, j)$, we have that P_{i-1}^v must use $\{g_5, s_2\}$ and P_j^h must use $\{g_5, t_1\}$. Similarly, P_{j-1}^h must use $\{g_9, s_3\}$ and P_i^v must use $\{g_9, g_6\}$. Since neither P_j^h nor P_i^v can use s_3 and P_j^h and P_i^v have no common edge, it follows that P_j^h and P_i^v cannot intersect in g_6. Thus, P_j^h must use $\{t_1, t_2\}$. A similar argument shows that P_i^v must use the edges $\{g_6, u_1\}$ and $\{u_1, u_2\}$. Due to symmetry, it follows that P_{i+1}^v must use the edge $\{u_3, u_2\}$. This is a contradiction as P_i^v and P_{i+1}^v cannot both use u_2. □

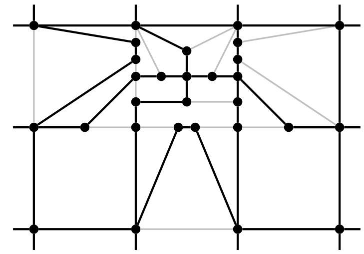
Consider the setting of Claim 11. Let G^+ be the plane graph obtained from G by applying the following modification in each connection face f_h of the clause gadget. A new vertex is inserted in f_h such that the new vertex is adjacent to the two original grid vertices, which are on the boundary of f_h and not on the boundary of f. The path consisting of these two new edges is called the new path of the connection face f_h.

Claim 12. *If G^+ contains an X-normal subdivision, which uses at most two new paths of the connection faces of the clause gadget, then G has an X-normal subdivision.*

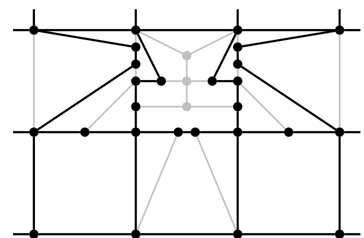
Proof. First, note that it suffices to show that there is an X_C-normal subdivision with the claimed property when no further modifications outside the faces in F are applied during the construction of G. Using the same notation as in the proof of Claim 11, Figure 5b) shows that there is an X_C-normal subdivision that pushes along (f_h, e_h, f) for all $h \in \{l, b\}$ simultaneously and Figure 5c) shows that there is an X_C-normal subdivision that pushes along (f_h, e_h, f) for all $h \in \{l, r\}$ simultaneously. Due to symmetry,



a) A clause gadget.



b) Pushing along (f_l, e_l, f) and (f_b, e_b, f).



c) Pushing along (f_l, e_l, f) and (f_r, e_r, f).

Figure 5: X_C-normal subdivisions in a positive clause gadget.

it follows that there is also an X_C-normal subdivision that pushes along (f_h, e_h, f) for all $h \in \{b, r\}$ simultaneously. □

The *negative clause gadget* is obtained by rotating the positive clause gadget around 180 degrees. Its *clause face* and its *connection faces*, which are the faces directly left, above, and right of the clause face, are defined accordingly. For a negative clause C, the set X_C consists of all original grid vertices that are on the boundary of the clause face. Claim 11 and Claim 12 hold analogously for the negative clause gadget.

4 Construction of G_ϕ

Consider an instance of the PLANAR MONOTONE 3-SAT PROBLEM, i.e., a monotone rectilinear representation $\mathcal{R}$ of a 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$. Define $n := |\mathcal{U}|$ and $m := |\mathcal{C}|$. Set $k := 8m + 2n + 5$. Denote by $\tilde{G}$ the $k \times k$ grid and consider $\tilde{G}$ together with its canonical embedding. Now, we describe how to add gadgets to $\tilde{G}$ in order to obtain G_ϕ.
First, the variable gadgets for the variables in $\mathcal{U}$ are added one after another in the middle row of $\tilde{G}$. More precisely, let $j_v := \lfloor \frac{1}{2} k \rfloor$ and let $F_\mathcal{U} := (f_1, e_2, f_2, \ldots, f_{k-1})$ be the straight face-sequence, where f_1 is the inner face of $\tilde{G}$ whose boundary contains the grid vertices $(1, j_v)$ and $(1, j_v + 1)$ and f_{k-1} is the inner face of $\tilde{G}$ whose boundary contains the grid vertices (k, j_v) and $(k, j_v + 1)$. Let $\mathcal{U} = \{x_1, \ldots, x_n\}$ and assume that $x_1, \ldots, x_n$ is the order in which the variables appear in the drawing $\mathcal{R}$. The variable gadgets

for $x_1, \ldots, x_n$ are placed, one after another, along $F_{\mathcal{U}}$ so that the left connection face of the variable gadget for x_1 is f_{m+3} and the right connection face of the variable gadget for x_n is $f_{7m+2n+2}$. In order to see that this is possible, set $d_i = 2(\deg^+(x_i) + \deg^-(x_i))$ for $i \in [n]$ and note that $\sum_{i\in[n]} \frac{1}{2}d_i$ counts each vertical line of $\mathcal{R}$ once and each clause in $\mathcal{R}$ touches exactly three vertical lines. Hence, $d_1 + \ldots + d_n = 6m$ and thus, exactly $(d_1 + 2) + \ldots + (d_n + 2) = 6m + 2n$ faces in a row are occupied by all variable gadgets together.

For an inner face f of G, the *boundary distance* of f is defined as the length of a shortest v,w-path in $\tilde{G}$ such that v is on the boundary of $\tilde{f}$ and $w \in V(B_{\text{outer}})$. Observe that every face of $\tilde{G}$ that belongs to a variable gadget has boundary distance at least $m + 2$. Denote by G_1 the graph obtained from $\tilde{G}$ by adding the variable gadgets in the described way. In the following, more faces of $\tilde{G}$ in G_1 are chosen for placing clause and wire gadgets. Using the rectilinear drawing $\mathcal{R}$, we choose distinct inner faces f_C of $\tilde{G}$ for each clause $C \in \mathcal{C}$ and a straight or almost straight face-sequence $\hat{F}_L$ in $\tilde{G}$ for each vertical line L in $\mathcal{R}$ such that the following two properties hold (see Figure 1 in Section 1.1).

(P1) If L is a vertical line in $\mathcal{R}$ that joins a variable $x \in \mathcal{U}$ to a positive (resp. negative) clause $C \in \mathcal{C}$, then the face-sequence $\hat{F}_L = (f_0, e_1, f_1, \ldots, f_d)$ satisfies: face f_0 is a top (resp. bottom) connection face of the variable gadget corresponding to x, face f_1 is the face directly above (resp. below) f_0, and face f_d is the clause face f_C.

(P2) For distinct vertical lines, the corresponding face-sequences do not have a common face, except possibly their last face, which is always a clause face.

(P3) For each clause $C \in \mathcal{C}$, there are exactly three face-sequences $\hat{F}_L$ with last face f_C.

(P4) Each face in $\hat{F}_L = (f_0, e_1, f_1, \ldots, f_d)$ has boundary distance at least $m + 2$. If $\hat{F}_L$ is an almost straight face-sequence, then $\hat{F}_L$ turns at f_h with $h \leq d - 2$.

Due to the properties of the rectilinear drawing $\mathcal{R}$ and the definition of k, it is easy to check that we can make such choices. Now, starting with G_1, for each face-sequence $\hat{F}_L = (f_0, e_1, \ldots, f_{d-1}, e_d, f_d)$ where L is a vertical line in $\mathcal{R}$, add a wire gadget along $(f_1, e_2, \ldots, e_{d-2}, f_{d-2})$. Observe that, for each clause $C \in \mathcal{C}$, the faces directly above, below, left, and right of f_C as well as f_C itself have not been modified so far. Next, for each positive (resp. negative) clause $C \in \mathcal{C}$, add a positive (resp. negative) clause gadget such that its clause face is f_C. Denote by G_ϕ the graph obtained in this way and define $X_\phi = \bigcup_{C\in\mathcal{C}} X_C$. Since G_1 is planar, it is easy to see that G_ϕ is planar. Also, observe that, in each face f of $\tilde{G}$ at most a constant number of vertices and edges has been added to construct the graph G_ϕ. Consequently, G_ϕ has size polynomial in k and also polynomial in the size of ϕ.

To state a key property of G_ϕ, we extend the face-sequences $\hat{F}_L$. Consider a vertical line L in $\mathcal{R}$ that joins a variable $x \in \mathcal{U}$ to a positive clause $C \in \mathcal{C}$ and denote by f the positive connection face of the variable gadget of x, which is the first face of $\hat{F}_L$. Let $(f_0, e_1, f_1, \ldots, e_d, f_d)$ be the straight face-sequence, where f_0 is the positive face of x and f_d is the face below f. Then, F_L denotes the face-sequence obtained by glueing $(f_0, e_1, \ldots, e_d, f_d)$ and $\hat{F}_L$ together. Moreover, for each variable $x \in \mathcal{U}$, denote by $\mathcal{F}_x^+$ the set of face-sequences that contains F_L for each vertical line L in $\mathcal{R}$ that joins x to a positive clause in $\mathcal{C}$. Similarly, define F_L for vertical lines L in $\mathcal{R}$ that join a variable to a negative clause as well as $\mathcal{F}_x^-$ for negative occurrences of x. The following claim holds due to Claim 7 and Claim 9.

Claim 13. *For each variable $x \in \mathcal{U}$, the pair (G_ϕ, X_ϕ) forces to push along each face-sequence in $\mathcal{F}_x^+$*

(resp. each face-sequence in $\mathcal{F}_x^-$) or (G_ϕ, X_ϕ) disqualifies.

5 Reduction

Using the notation of the previous sections, we now show that the formula ϕ is satisfiable if and only if the graph G_ϕ contains a subdivision of a $k \times k$ grid.

5.1 If ϕ is satisfiable

Assume that ϕ is satisfiable. The aim of this subsection is to show that G_ϕ then contains a subdivision of a $k \times k$ grid. Let $T : \mathcal{U} \to \{\text{TRUE}, \text{FALSE}\}$ be a satisfying assignment of ϕ and denote by $E_\mathcal{U}$ the set of edges of $\tilde{G}$ that were deleted due to the decision part of each variable gadget when constructing G_ϕ, i.e., $E_\mathcal{U} = \{e_x \colon x \in \mathcal{U}\}$. Recall that, for $e = \{x, y\} \in E(\tilde{G}) \setminus E_\mathcal{U}$, the path P_e is the final x,y-path in G_ϕ that replaces e (if $e \in E(G_\phi)$, then $P_e = e$). Let $\tilde{G}_s$ be the graph obtained from $\tilde{G}$ by replacing each edge $e \in E(\tilde{G}) \setminus E_\mathcal{U}$ by P_e. Clearly, $\tilde{G}_s$ is a subdivision of $\tilde{G}$ and the graph $H' := \tilde{G}_s - E_\mathcal{U}$ is a subgraph of G_ϕ.
For each variable $x \in \mathcal{U}$ with $T(x) = \text{FALSE}$, the edge e_x of $\tilde{G}_s$ can be replaced by the positive edges of x. In order to maintain a subdivision of the $k \times k$ grid, we modify the graph such that it pushes along each face-sequence in $\mathcal{F}_x^+$. This works due to Claim 8, Claim 10, and Claim 12. Indeed, for each positive clause C there is at least one variable x used in C with $T(x) \neq \text{FALSE}$, which implies that there is a face-sequence F_L with last face f_C that relaxes. Hence, there are no problems in the clause faces. Analogously, we can replace each edge e_x of $\tilde{G}_s$ with $T(x) = \text{TRUE}$ and while doing so there are no problems in the faces of negative clauses.

5.2 If G_ϕ contains a subdivision of a $k \times k$ grid

The aim of this section is to prove that, if G_ϕ contains a subdivision of a $k \times k$ grid, then ϕ is satisfiable. The main part of the proof is the next lemma.

Lemma 14. *If G_ϕ contains a subgraph H that is isomorphic to a subdivision of a $k \times k$ grid, then H is X_ϕ-normal.*

Before proving the previous lemma, we use it to argue that ϕ is satisfiable. Assume that G_ϕ contains a subgraph H that is isomorphic to a subdivision of $k \times k$ grid. Due to the previous lemma, H must be X_ϕ-normal. In order to define a truth assignment $T : \mathcal{U} \to \{\text{TRUE}, \text{FALSE}\}$ for ϕ, consider a variable $x \in \mathcal{U}$. According to Claim 7a), the decision path of x in H uses either both positive or both negative edges of the decision gadget corresponding to x. Set $T(x) = \text{TRUE}$ if and only if the decision path of x uses the negative edges of x. For a contradiction, assume that there is a positive clause $C \in \mathcal{C}$ that is not satisfied by the assignment T; the following is easy to adjust for a negative clause. Let x be an arbitrary variable used in C and denote by L a vertical line in $\mathcal{R}$ that joins x to C. Then $T(x) = \text{FALSE}$ and, hence, the decision path of x uses the positive edges of x. By Claim 13, the pair (G_ϕ, X_ϕ) forces to push along each face-sequence in $\mathcal{F}_x^+$ and therefore H pushes along F_L. As, in $\mathcal{R}$, there are exactly three

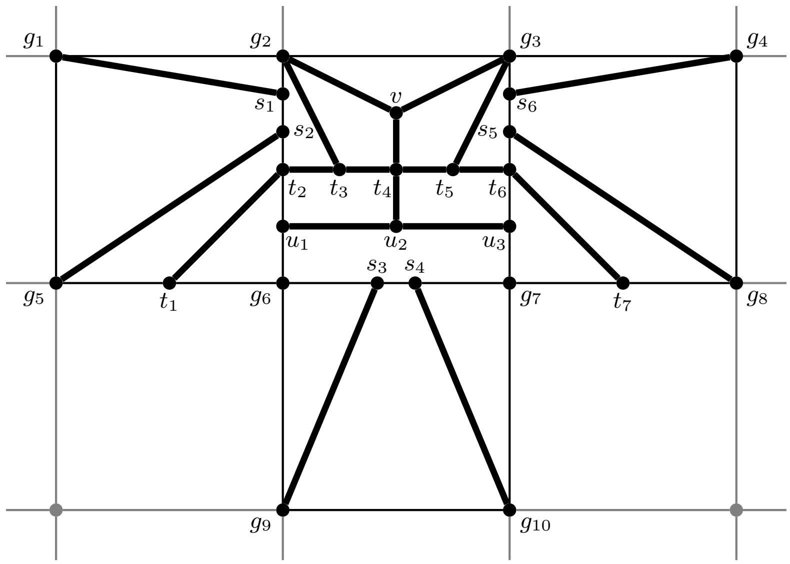


Figure 6: A positive clause gadget. The face of $\tilde{G}$ that is bounded by the cycle (g_2, g_3, g_7, g_6) is the clause face. The vertices g_h with $h \in [10]$ are original grid vertices.

vertical lines that touch the clause C, there are three distinct face-sequences F_L, which have no common face except for the last face f_C due to Property (P2) and such that H pushes along F_L. This contradicts Claim 11 and, hence, C must be satisfied. Consequently, ϕ is a satisfying assignment.

5.2.1 Proof of Lemma 14

Throughout this subsection, assume that G_ϕ contains a subgraph H, which is isomorphic to a subdivision of a $k \times k$ grid. Whenever a positive clause gadget is considered, we use the vertex names introduced in Figure 6. Recall that $X_C = \{g_2, g_3, g_6, g_7\}$ for all positive clauses $C \in \mathcal{C}$ and $X_\phi = \bigcup_{C \in \mathcal{C}} X_C$. To prove Lemma 14, we have to show that

(i) B_{outer} bounds the outer face of H and

(ii) there is a vertex map f_V for H with $f_V(w) = w$ for all $w \in V(\tilde{G}) \setminus X_\phi$.

To introduce some more notation, consider a positive clause $C \in \mathcal{C}$. Let

$$V_4^C := \{g_2, g_3, g_6, g_7, t_2, t_4, t_6\},$$

which is the set of vertices of degree at least 4 in G_ϕ that belong to the clause face of the clause gadget corresponding to C. Define V_4^C analogously for negative clauses $C \in \mathcal{C}$. For a graph G and $d \in \mathbb{N}$, let $V_d^G := \{v \in V(G) \colon \deg_G(v) \geq d\}$. By construction,

$$\left|V_4^{G_\phi}\right| = (k-2)^2 + 3m, \qquad \left|V_4^H\right| = (k-2)^2, \quad \text{and} \quad \left|V_3^H \setminus V_4^H\right| = 4(k-2), \tag{1}$$

since each clause gadget contains 3 vertices v with $\deg_{G_\phi}(v) \geq 4$ that are not original grid vertices, namely t_2, t_4, and t_6.

Before we proceed with the proof, we sketch the main idea in the next remark.

Remark 1 (sketch of the proof of Lemma 14). *In order to prove Lemma 14, we first note that few vertices of degree at least 4 are created in order to obtain G_ϕ from $\tilde{G}$ and these vertices are in the clause faces. Further, each clause face contains at most 5 vertices with degree 4 in H (Claim 15). Thus, most of the intersection vertices of H with degree 4 must be original grid vertices. As k is chosen so that Property (P4) holds and using the aforementioned fact, we can show that there must be vertices of the outer face of H, which are vertices of the outer face of $\tilde{G}$ (Claim 17), and no vertex of the outer face of H belongs to a clause face (Claim 18). The previous fact allows us to prove that each clause face in fact contains at most 4 vertices with degree 4 in H (Claim 19). We conclude that the outer face of H coincides with the outer face of G_ϕ and, finally, a separation argument implies that H is X_ϕ-normal.*

Claim 15. *For every clause $C \in \mathcal{C}$ we have $\left|V_4^C \cap V_4^H\right| \le 5$.*

Proof. Consider a positive clause $C \in \mathcal{C}$; the following argument is easy to adjust for a negative clause. First, note that it suffices to show that

$$\{t_2, g_6\} \nsubseteq V_4^H \qquad \text{and} \qquad \{t_6, g_7\} \nsubseteq V_4^H. \tag{2}$$

Due to symmetry of the clause gadget we only need to show that $\{t_2, g_6\} \nsubseteq V_4^H$.
Towards a contradiction, assume that $\deg_H(t_2) = \deg_H(g_6) = 4$. Denote by P_1 and P_1' the grid-paths of H that intersect in t_2. Due to Observation 5, we may assume that P_1 contains the subpath (s_2, t_2, u_1) and that P_1' contains the subpath (t_1, t_2, t_3). Similarly, let P_2 and P_2' be the grid-paths of H that intersect in g_6 and assume that $(u_1, g_6, g_9) \subseteq P_2$ and $(t_1, g_6, s_3) \subseteq P_2'$.
Case 1: P_1 and P_2 are of the same type. Since P_1 and P_2 both use u_1, we have $P_1 = P_2$ as otherwise Observation 3 was violated. Therefore, P_1' and P_2' are of the same type. As P_1' and P_2' both use t_1, again Observation 3 implies that $P_1' = P_2'$. But then, the paths P_1 and P_1' have two vertices in common, namely t_2 and g_6, which contradicts Observation 3.
Case 2: P_1 and P_2 are of distinct types. Then, u_1 is an intersection vertex of H with $\deg_H(u_1) \le 3$. Moreover, u_1 is adjacent to the intersection vertices t_2 and g_6, which satisfy $\deg_H(t_2) = \deg_H(g_6) = 4$. This is a contradiction, as any vertex of degree at most 3 in a $k' \times k'$ grid with $k' \ge 3$ is adjacent to at most one vertex of degree 4. □

Denote by B_H the set of vertices of H that are on the boundary of the outer face of H. For each clause $C \in \mathcal{C}$, let V^C be the set of vertices that belong to the clause face f_C: namely, vertices that are on the boundary of f_C, embedded inside f_C or inserted to subdivide an edge on the boundary of f_C.
Throughout this section, define $W := \bigcup_{C \in \mathcal{C}} V_4^C$ and $\overline{W} := V_4^{G_\phi} \setminus W$. The next observation follows immediately from Claim 15.

Observation 16. *We have $|V_4^H \cap W| \le 5m$ and $|V_4^H \cap \overline{W}| \ge (k-2)^2 - 5m$.*

Claim 17. *The set B_H contains at least one vertex from B_{outer}.*

Proof. We first prove that

$$\left|B_H \cap \overline{W}\right| \le m. \tag{3}$$

Towards a contradiction, assume that B_H contains more than m vertices from $\overline{W}$, which are original grid vertices, that have degree 4 in G_ϕ and that do not belong to a clause gadget. Since $|\overline{W}| = (k-2)^2 - 4m$, we have $|V_4^H \cap \overline{W}| \leq |\overline{W}| - |B_H \cap \overline{W}| < (k-2)^2 - 5m$, which contradicts Observation 16. Thus, (3) is indeed satisfied.
Next, suppose that $B_H \cap V(B_{\text{outer}}) = \emptyset$. Let G' be the graph obtained from G_ϕ by deleting all vertices in $V_4^{G_\phi}$ and all vertices in $V(B_{\text{outer}})$. It is easy to see that each component of G' contains at most 3 vertices. So, at least every fourth vertex of B_H belongs to $V_4^{G_\phi}$. Since the boundary of the outer face of a $k \times k$ grid contains $4(k-1)$ vertices, B_H contains at least $4(k-1)$ vertices. Hence, $|B_H \cap V_4^{G_\phi}| \geq k-1$. Now, (1) implies that

$$(k-2)^2 = \left|V_4^H\right| \leq \left|V_4^{G_\phi}\right| - \left|B_H \cap V_4^{G_\phi}\right| \leq (k-2)^2 + 3m - (k-1),$$

which is equivalent to $k - 1 \leq 3m$ and contradicts the definition of k.

□

Claim 18. *For each $C \in \mathcal{C}$, we have $B_H \cap V^C = \emptyset$.*

Proof. For a contradiction, assume that there is a clause $C \in \mathcal{C}$ with $B_H \cap V^C \neq \emptyset$. Claim 17 implies that there is a path $P \subseteq G_\phi$ with $V(P) \subseteq B_H$ that starts in a vertex $v \in V^C$ and ends in a vertex $w \in V(B_{\text{outer}})$. Recall Property (P4), which implies that every face of $\tilde{G}$ that was modified when constructing G_ϕ has boundary distance at least $m+2$. Hence, f_C has boundary distance at least $m+2$ and P contains at least $m+2$ original grid vertices of G_ϕ that are not in W and at least $m+1$ of these vertices have degree 4 in G_ϕ. Therefore, B_H contains at least $m+1$ vertices from $\overline{W}$, which is a contradiction to (3). □

The next claim states that each clause face contains at most 4 vertices of H of degree 4.

Claim 19. *For each clause $C \in \mathcal{C}$, we have $\left|V_4^C \cap V_4^H\right| \leq 4$.*

Proof. Without loss of generality, consider a positive clause $C \in \mathcal{C}$. The following properties are used in the remaining proof without further mentioning them:

- Any two distinct grid paths of H of the same type do not have a common vertex and any two grid paths of H of different types have precisely one common vertex (Observation 3).
- Grid-paths of H cannot end in a vertex of V^C, i.e., all intersection vertices of H that lie in V^C have degree 4 with respect to H (Claim 18).
- In the drawing of G, every horizontal path crosses each vertical path of H at their unique common vertex (Observation 5).

In order to prove the claim, we basically check that there are no 5 vertices in V_4^C that belong to V_4^H simultaneously. First, we show that

$$\{t_2, t_4, t_6\} \not\subseteq V_4^H. \tag{4}$$

For the sake of contradiction, we assume that $\deg_H(t_2) = \deg_H(t_4) = \deg_H(t_6) = 4$. Let P and P' be the grid-paths of H that intersect in t_4. Without loss of generality, we can assume that $(v, t_4, u_2) \subseteq P$

and $(t_3, t_4, t_5) \subseteq P'$. Analogously, there are grid-paths P_1, P_1', P_2, and P_2' of H with $(s_2, t_2, u_1) \subseteq P_1$, $(t_1, t_2, t_3) \subseteq P_1'$, $(s_5, t_6, u_3) \subseteq P_2$, and $(t_5, t_6, t_7) \subseteq P_2'$. Now, t_3 and t_5 are not intersection vertices of H and, hence, $P' = P_1' = P_2'$. Then, P, P_1, and P_2 are distinct paths of the same type. Moreover, the path P cannot end in u_2 and, hence, it contains either the edge $\{u_2, u_1\}$ or the edge $\{u_2, u_3\}$. Therefore, P intersects P_1 or P_2, which is a contradiction.

Due to (2) and (4), it suffices to show the following three items in order to finish the proof.

a) $\{g_2, g_3, g_7, t_2, t_4\} \nsubseteq V_4^H$
b) $\{g_2, g_3, g_6, t_4, t_6\} \nsubseteq V_4^H$
c) $\{g_2, g_3, g_6, g_7, t_4\} \nsubseteq V_4^H$

We first prove a). For the sake of contradiction, we assume that the degree of each vertex g_2, g_3, g_7, t_2, and t_4 in H is four. Then, there are grid-paths P, P', P_1, P_2 and P_2' of H with $(v, t_4, u_2) \subseteq P$, $(t_1, t_2, t_3, t_4, t_5) \subseteq P'$, $(s_2, t_2, u_1) \subseteq P_1$, $(u_3, g_7, g_{10}) \subseteq P_2$ and $(s_4, g_7, t_7) \subseteq P_2'$. Further, there are grid-paths Q_1 and Q_1' which intersect in g_2 and grid-paths Q_2, Q_2' which intersect in g_3.

Since $P_1 \neq P$ as well as that P_1 and P are of the same type, P_1 uses the edge $\{u_1, g_6\}$ and P uses the edge $\{u_2, u_3\}$. Thus, $P = P_2$. The path P uses the edge $\{v, g_2\}$ or the edge $\{v, g_3\}$.

Case 1: P uses $\{v, g_3\}$. Since P and P' intersect already in t_4, the path P' must use the edge $\{t_5, t_6\}$. Moreover, P' and P_2' are distinct and of the same type as they both intersect P. Hence, P_2' uses the edge $\{t_7, g_8\}$ and $(t_6, s_5, s_6) \subseteq P'$. Additionally, P' cannot use g_3 as P uses g_3 and P' and P already intersect in t_4. Thus, P' uses g_4 and we have $(t_5, t_6, s_5, s_6, g_4) \subseteq P'$. Further, $P \in \{Q_2, Q_2'\}$, say $P = Q_2$. Moreover, the paths Q_2' and P' are distinct and of the same type. Therefore, the edges $\{g_3, g_4\}$, $\{g_3, s_6\}$, and $\{g_3, t_5\}$ are not contained in Q_2' and neither in Q_2. Consequently, these edges do not belong to H and $\deg_H(g_3) < 4$, which is a contradiction.

Case 2: P uses g_2. Then, $P \in \{Q_1, Q_1'\}$, say $P = Q_1$. Moreover, P' uses the edge $\{t_1, g_5\}$ since P_1 uses g_6 and intersects P' already in t_2. Then, $(s_2, s_1, g_1) \subseteq P_1$, since P_1 and P' already intersect in t_2 and since P and P_1 are distinct paths of the same type, which implies that P_1 cannot use g_2. Now, Q_1' and P_1 are of different types. Hence, Q_1' and P_1 can only intersect in vertices with degree ≥ 4 in G_ϕ and, hence, Q_1' uses neither s_1 nor t_3. Moreover, P uses neither s_1 nor t_3, since P and P_1 are of the same type and P and P' already intersect in t_4. Thus, Q_1' uses the edge $\{g_2, g_3\}$ and $Q_1' \in \{Q_2, Q_2'\}$, say $Q_1' = Q_2'$. Then, P', P_2', and Q_1' are distinct paths of the same type. Therefore, $(g_7, t_7, g_8) \subseteq P_2'$ and $(t_5, t_6, s_5, s_6, g_4) \subseteq P'$. Since Q_1' must use t_5, s_6, or g_4, the paths Q_1' and P' have a common vertex and are of the same type, which is a contradiction.

Statement a) is equivalent to b) due to symmetry of the clause gadget.

We now prove c). Towards a contradiction, assume that g_2, g_3, g_6, g_7, and t_4 are vertices of degree 4 in H. Then, there are grid-paths P, P', P_1, P_1', P_2, and P_2' of H with $(v, t_4, u_2) \subseteq P$, $(t_3, t_4, t_5) \subseteq P'$, $(u_1, g_6, g_9) \subseteq P_1$, $(t_1, g_6, s_3) \subseteq P_1'$, $(u_3, g_7, g_{10}) \subseteq P_2$, and $(s_4, g_7, t_7) \subseteq P_2'$. Now, $P_1' = P_2'$ since P_1' cannot use g_9 and P_2' cannot use g_{10}. Moreover, P uses u_1 or u_3. Due to symmetry, we may assume that P uses u_1 and, hence, $P = P_1$. Let Q_1, Q_1' and Q_2, Q_2' be the grid-paths of H that intersect in g_2 and g_3, respectively. The path P uses either the edge $\{v, g_2\}$ or the edge $\{v, g_3\}$.

Case 1: P uses g_2. Then $P \in \{Q_1, Q_1'\}$, say $P = Q_1$. Similar to the arguments used in the proof of Statement a), one can argue that $(t_3, t_2, s_2, s_1, g_1) \subseteq P'$ and $(t_1, g_5) \subseteq P_1'$. Since Q_1' must use g_1, s_1,

or t_3, the paths Q'_1 and P' have a common vertex and are of the same type, which is a contradiction.
Case 2: P uses g_3. Then, $(t_5, t_6) \subseteq P'$ and $(u_3, t_6) \subseteq P_2$. As P_2 and P' are of different types, they intersect in t_6. Hence, $t_6 \in V_4^H$, which contradicts (2). □

Claim 19 implies that $|V_4^H \cap W| \leq 4m$. By construction, we have $|\overline{W}| = (k-2)^2 - 4m$. Thus, $\overline{W} \subseteq V_4^H$ as otherwise $|V_4^H| < |V_4^H \cap W| + |\overline{W}| = (k-2)^2$, which violates (1).

Observation 20. $V_4^{G_\phi} \setminus W \subseteq V_4^H$.

Denote by $\tilde{P}_1^v, \ldots, \tilde{P}_k^v$ and $\tilde{P}_1^h, \ldots, \tilde{P}_k^h$ the vertical and horizontal paths of $\tilde{G}$. For $i \in [k]$, define a set S_i^v as follows. If none of the vertices in $\tilde{P}_i^v$ is on the boundary of a face f_C with $C \in \mathcal{C}$, let $S_i^v = V(\tilde{P}_i^v)$. Otherwise, there is exactly one clause $C \in \mathcal{C}$ such that $\tilde{P}_i^v$ contains a vertex on the boundary of f_C due to the construction of G_ϕ. Assume that C is a positive clause. If, with reference to the vertex names in Figure 6, $\{g_2, g_6\} \subseteq V(\tilde{P}_i^v)$, then let $S_i^v = (V(\tilde{P}_i^v) \setminus \{g_2, g_6\}) \cup \{g_1, g_5\}$, and otherwise let $S_i^v = (V(\tilde{P}_i^v) \setminus \{g_3, g_7\}) \cup \{g_4, g_8\}$. If C is a negative clause, define S_i^v analogously. Moreover, for $j \in [k]$, define $S_j^h = V(P_j^h)$. The next claim is easy to verify.

Claim 21. *For each $i \in [k]$, the set S_i^v separates $V(\tilde{P}_1^v)$ from $V(\tilde{P}_k^v)$ in G_ϕ and, for each $j \in [k]$, the set S_j^h separates $V(\tilde{P}_1^h)$ from $V(\tilde{P}_k^h)$ in G_ϕ.*

Now, we have all technical details to prove that H is X_ϕ-normal.
Proof of Lemma 14. Towards a contradiction, assume that $B_H \neq V(B_{\text{outer}})$. Due to Claim 17, the set B_H contains at least one vertex from B_{outer}. Hence, there is an edge $e = \{v, w\} \in E(H)$ with $v \in V(B_{\text{outer}})$ and $w \notin V(B_{\text{outer}})$. Then, $w \notin V_4^H$ and (P4) implies that $w \in V_4^{G_\phi} \setminus W$, which together is a contradiction to Observation 20. Consequently, $B_H = V(B_{\text{outer}})$.
Let f_V be a vertex map for H. Since $B_H = V(B_{\text{outer}})$, we have

$$\{f_V((1,1)), f_V((1,k)), f_V((k,1)), f_V((k,k))\} \subseteq \{(1,1), (1,k), (k,1), (k,k)\}.$$

Due to symmetry of the $k \times k$ grid, we may assume without loss of generality that $f_V((i,j)) = (i,j)$ for all $(i,j) \in B_{\text{outer}}$. Denote by $P_1^v, \ldots, P_k^v$ and $P_1^h, \ldots, P_k^h$ the vertical and horizontal paths of H that correspond to the vertex map f_V. Next, we consider the original grid vertices of G_ϕ with degree ≥ 4 that do not belong to a clause gadget. Fix $(i,j) \in V_4^{G_\phi} \setminus W$. Recall the sets S_j^h and S_i^v defined above and note that S_i^v contains (i,j). There are exactly k horizontal paths in H, which are pairwise vertex-disjoint, and each of them joins a vertex in $V(\tilde{P}_1^v)$ to a vertex in $V(\tilde{P}_k^v)$. Furthermore, the set S_i^v contains exactly k vertices. Hence, each horizontal path of H uses exactly one vertex from S_i^v, due to Claim 21. Since G_ϕ is planar and the horizontal paths do not intersect, the path P_j^h must use (i,j). Analogously, it follows that P_i^v must use (i,j). Consequently $f_V((i,j)) = (i,j)$ for all $(i,j) \in V_4^{G_\phi} \setminus W$. Observing that $V(\tilde{G}) \setminus X_\phi \subseteq V_4^{G_\phi} \setminus W$ implies that H is X_ϕ-normal. □

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